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ELLIPSOIDAL CORRECTIONS FOR GEOID UNDULATION COMPUTATIONS

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by

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Abstract

The computation of accurate geoid undulations is usually done combining potential coefficient information and terrestrial gravity data in a cap surrounding the computation point. In doing this a spherical approximation is made that can cause errors that are investigated in this paper. The equations dealing with ellipsoidal corrections developed by Lelgemann and by Moritz are used to develop a computational procedure considering the ellipsoid as a reference surface. Terms in the resulting expression for the geoid undulation are identified as ellipsoidal correction terms. These equations have been developed for the case where the Stokes function is used, and for the case where the modified Stokes function is used. For a cap of 20° the correction can reach -33cm.

Ellipsoidal corrections were also computed for the Marsh/Chang geoids. These corrections reach -45cm for a cap size of 20°.

Global maps are given showing the distribution of the corrections so that more accurate geoid undulations can be found.

Foreword

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1. Introduction

Geoid undulations can be computed from Stokes' equation given a global set of gravity anomalies. The Stokes' equation is usually given with spherical approximation (Heiskanen and Moritz, 1967, p. 94). The error of the spherical approximation has been investigated by several authors who have sought a solution of the boundary value problem with the ellipsoid as a reference surface. Lelgemann (1970) developed techniques to compute corrections to the undulations computed from the Stokes' equation. He showed that the root mean square correction was ± 0.2 m with a maximum correction of about 0.6 m. A somewhat revised approach to this approximation problem is discussed by Moritz (1980).

Today, however, geoid undulations are not usually computed through a global integration procedures. Instead potential coefficient information is combined with gravity anomaly information in a cap surroudning the computation point (Rapp and Rummel, 1975, Marsh and Chang, 1976, Rapp, 1980). For this procedure we therefore cannot use directly the correction equations developed by Lelgemann; instead we must develop new equations.

2. The Theory

We start with the following representation of the earth's gravitational disturbing potential:

$$(1)\quad T(r, \bar{\varphi}, \lambda) = \frac{kM}{r} \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm}(\sin\bar{\varphi})$$

where:

- kM geocentric gravitational constant;
- r geocentric radius;
- a earth equatorial radius used in potential coefficient determinations;
- C_{nm}, S_{nm} potential coefficients. (The $C_{2,0}$, $C_{4,0}$, $C_{6,0}$ etc. are referenced to a defined reference ellipsoid.)
- $P_{nm}(\sin\bar{\varphi})$ associated Legendre functions, as a function of geocentric latitude $\bar{\varphi}$.

The disturbing potential can be evaluated on the ellipsoid (assume no external masses) by letting $r = r_E$ the geocentric radius to a point on the ellipsoid. We now can compare this to the following representation of T on the ellipsoid (Moritz, 1980, p. 320):

$$(2)\quad T(\varphi, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)\, P_{nm}(\sin\varphi)$$

where A_{nm}, B_{nm} are coefficients to be determined; φ is the geodetic latitude. We need to find the relationship between the A, B coefficients and the C, S coefficients noting that at the same point, equation (1) and (2) must yield the

same potential. We write (in a linear approximation).

$$(3)\quad T(\varphi, \lambda, A, B,) = T(\bar{\varphi}, \lambda, A, B) + \frac{\partial T}{\partial \varphi}(\varphi - \bar{\varphi})$$

$$= T(\bar{\varphi}, \lambda, A, B) +$$

$$\sum_{n=2}^{\infty} \sum_{m=0}^{n} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) \left. \frac{d P_{nm}(\sin\varphi)}{d\varphi} \right|_{\bar{\varphi}} (\varphi - \bar{\varphi})$$

Now from geometric geodesy we know:

$$(4)\quad (\varphi - \bar{\varphi}) \approx e^2 \sin\bar{\varphi} \cos\bar{\varphi}$$

From Moritz (ibid, 39-46,) we can write:

$$(5)\quad \sin\bar{\varphi} \cos\bar{\varphi} \frac{d P_{nm}}{d\psi} = a_{nm} P_{n+2,m} + b_{nm} P_{nm} + c_{nm} P_{n-2,m}$$

where:

$$(6)\quad \begin{aligned} a_{nm} &= \frac{-n(n - m + 1)(n - m + 2)}{(2n + 1)(2n + 3)} \\ b_{nm} &= \frac{n^2 - 3m^2 + n}{(2n + 3)(2n - 1)} \\ c_{nm} &= \frac{(n + 1)(n + m)(n + m - 1)}{(2n + 1)(2n - 1)} \end{aligned}$$

We now use (4) and (5) in (3) to write:

$$(7)\quad T(\varphi, \lambda, A, B) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) P_{nm}(\sin\bar{\varphi})$$

$$+ e^2 \sum_{n=2}^{\infty} \sum_{m=0}^{n} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)(a_{nm} P_{n+2,m} + b_{nm} P_{nm} + c_{nm} P_{n-2,m})$$

To simplify (7) we use the following relationships (ibid, eq. 39-48):

$$(8)\quad \begin{aligned} \sum_{n=2}^{\infty} a_{nm} \begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} P_{n-2,m} &= \sum_{n=0}^{\infty} a_{n+2,m} \begin{Bmatrix} A_{n+2,m} \\ B_{n+2,m} \end{Bmatrix} P_{nm} \\ \sum_{n=2}^{\infty} a_{nm} \begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} P_{n+2,m} &= \sum_{n=4}^{\infty} a_{n-2,m} \begin{Bmatrix} A_{n-2,m} \\ B_{n-2,m} \end{Bmatrix} P_{nm} \end{aligned}$$

Now we can write (7) as follows:

$$(9)\quad T(\varphi, \lambda, A, B) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \Big[(A_{nm} + e^2 K_{nm}) \cos m\lambda$$

$$+ (B_{nm} + e^2 L_{nm}) \sin m\lambda \Big] P_{nm}(\sin\bar{\varphi})$$

where:

$$(10)\quad \begin{aligned} K_{nm} &= a_{n-2,m} A_{n-2,m} + b_{nm} A_{nm} + c_{n+2,m} A_{n+2,m} \\ L_{nm} &= a_{n-2,m} B_{n-2,m} + b_{nm} B_{nm} + c_{n+2,m} B_{n+2,m} \end{aligned}$$

(Note that the summation on n has started from zero as a consequence of the relationships in (8).) We now equate (9) and (1) with $r = r_E$ to find

$$(11)\quad \frac{kM}{r_E}\left(\frac{a}{r_E}\right)^n \begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \begin{Bmatrix} A_{nm} + e^2 K_{nm} \\ B_{nm} + e^2 L_{nm} \end{Bmatrix}$$

The A_{nm}, B_{nm} coefficients are a function of φ since r_E is a function of φ. In a spherical approximation (11) becomes

$$(12)\quad \frac{kM}{R}\left(\frac{a}{R}\right)^n \begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix}$$

We next consider the ellipsoidal form of the Stokes' equation as given by Moritz (ibid, eq. (49-26)):

$$(13)\quad N_E = \frac{R}{4\pi\gamma}\iint_\sigma (\Delta g - e^2\,\Delta g')\,S(\psi)\,d\sigma + e^2\left(\frac{1}{4} - \frac{3}{4}\sin^2\varphi\right)N$$

Here we have assumed the solution of the geoid boundary value problem as opposed to the Molodensky problem as discussed by Moritz. In (13) we have: γ, average value of normal gravity over the ellipsoid; N, approximate value of the geoid undulation. In addition (Moritz, ibid., 39-80):

$$(14)\quad \Delta g' = \frac{1}{R}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(G_{nm}\cos m\lambda + H_{nm}\sin m\lambda)\,P_{nm}(\sin\varphi)$$

where:

$$(18)\quad \begin{Bmatrix} G_{nm} \\ H_{nm} \end{Bmatrix} = \varkappa_{nm}\begin{Bmatrix} A_{n-2,m} \\ B_{n-2,m} \end{Bmatrix} + \lambda_{nm}\begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} + \mu_{nm}\begin{Bmatrix} A_{n+2,m} \\ B_{n+2,m} \end{Bmatrix}$$

with

$$(19)\quad \begin{aligned} \varkappa_{nm} &= \frac{-3(n-3)(n-m-1)(n-m)}{2(2n-3)(2n-1)} \\ \lambda_{nm} &= \frac{n^3 - 3m^2n - 9n^2 - 6m^2 - 10n^2 + 9}{3(2n+3)(2n-1)} \\ \mu_{nm} &= \frac{-(3n+5)(n+m+2)(n+m+1)}{2(2n+5)(2n+3)} \end{aligned}$$

Although Moritz starts the summation from $n = 2$ in (14), the formality of the reduction leading to (14) indicates a starting summation from $n = 0$. We also have $d\sigma = \cos\varphi\, d\varphi\, d\lambda$ (Lelgemann, 1970).

Now consider the evaluation of (13) with the Δg values considered in a cap σ_c surrounding the computation point, and in the remaining area of the sphere. We have

$$(20)\quad N_E = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}(\Delta g - e^2\,\Delta g')\,S(\psi)\,d\sigma + \frac{R}{4\pi\gamma}\int_{\psi_0}^{\pi}\int_0^{2\pi}(\Delta g - e^2\,\Delta g')\,S(\psi)\,d\sigma + \Delta N$$

where

$$(20a)\quad \Delta N = e^2\left(\frac{1}{4} - \frac{3}{4}\sin^2\varphi\right)N$$

In (20) ψ_0 is the spherical cap radius surrounding the computation point.

We introduce the new function $\bar{S}(\psi)$ (Heiskanen and Moritz, 1967, p. 259):

$$\text{(21)} \quad \bar{S}(\psi) = \begin{cases} 0 & \text{if } 0 \le \psi < \psi_0 \\ S(\psi) & \end{cases}$$

Equation (20) then becomes:

$$\text{(22)} \quad N_\varepsilon = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} (\Delta g - e^2 \Delta g')\, S(\psi)\, d\sigma + \frac{R}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} (\Delta g - e^2 \Delta g')\, \bar{S}(\psi)\, d\sigma + \Delta N$$

We now write (Moritz, ibid, p.326):

$$\text{(23)} \quad \Delta g = \Delta g^{o} + e^2 \Delta g'$$

where

$$\text{(24)} \quad \Delta g^{o} = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{(n-1)}{R} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)\, P_{nm}(\sin\varphi) .$$

Using (23) for $\Delta g'$ in (22) we have:

$$\text{(25)} \quad N_\varepsilon = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} (\Delta g - e^2 \Delta g')\, S(\psi)\, d\sigma + \frac{R}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g^{o}\, \bar{S}(\psi)\, d\sigma + \Delta N$$

Following Heiskanen and Moritz (1967, p.259) we introduce the Molodenskii coefficients Q_n:

$$\text{(26)} \quad \bar{S}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n P_n(\cos\psi)$$

so that the second integral in equation (25) can be written as:

$$\text{(27)} \quad \frac{R}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g^{o}\, \bar{S}(\psi)\, d\sigma = \frac{R}{8\pi\gamma}\sum_{n=0}^{\infty} (2n+1) Q_n \int_0^{\pi}\int_0^{2\pi} \Delta g^{o} P_n(\cos\psi)\, d\sigma$$
$$= \frac{R}{2\gamma}\sum_{n=2}^{\infty} Q_n \Delta g_n^{o}$$

where Δg^{o} would be obtained from (24).

If we now use (24) for Δg_n^{o} and use (11) for the coefficient relationships we can express (25) as:

$$\text{(28)} \quad N_\varepsilon = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} (\Delta g - e^2 \Delta g')\, S(\psi)\, d\sigma$$
$$+ \frac{kM}{2 r_\varepsilon \gamma} \sum_{n}^{\infty} Q_n (n-1) \left(\frac{a}{r_\varepsilon}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda$$
$$+ S_{nm} \sin m\lambda)\, P_{nm}(\sin\varphi)$$
$$- \frac{e^2}{2\gamma} \sum_{n}^{\infty} Q_n (n-1) \sum_{m=0}^{n} (K_{nm} \cos m\lambda + L_{nm} \sin m\lambda) P_{nm}(\sin\varphi) + \Delta N$$

The summation on n is from 2 if we ignore the zero and first degree terms that have been eliminated through the Stokes' equation. Note that the correction terms involving K and L arise from the fact that the associated Legendre

functions are evaluated at the geodetic latitude instead of the geocentric latitude. If the evaluation is done at the geocentric latitude the correction terms will not be necessary. We have:

$$(29)\quad N_E = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} (\Delta g - e^2 \Delta g')\, S(\psi)\, d\sigma$$

$$+ \frac{kM}{2 r_E \gamma}\sum_{n=2}^{\infty} Q_n (n-1) \left(\frac{a}{r_E}\right)^n \sum_{m=0}^{n} (C_{nm}\cos m\lambda$$

$$+ S_{nm}\sin m\lambda)\, P_{nm}(\sin\bar{\varphi}) + \Delta N$$

Now consider the integral in (29) involving $e^2 \Delta g'$. We write:

$$(30)\quad \frac{Re^2}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} \Delta g'\, S(\psi)\, d\sigma = \frac{Re^2}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g'\, S(\psi)\, d\sigma - \frac{Re^2}{4\pi\gamma}\int_{\psi_0}^{\pi}\int_0^{2\pi} \Delta g'\, S(\psi)\, d\sigma$$

The first integral on the right hand side of (30) can be written as (Moritz, ibid p. 426):

$$(31)\quad \frac{Re^2}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g'\, S(\psi)\, d\sigma = \frac{e^2}{\gamma}\sum_{n=2}^{\infty}\frac{1}{(n-1)}\sum_{m=0}^{n} (G_{nm}\cos m\lambda$$

$$+ H_{nm}\sin m\lambda)\, P_{nm}(\sin\bar{\varphi})$$

Note the summation starts from $n = 2$ because of properties of the Stokes' function. Recall equation (14), however, where $\Delta g'$ contains zero and first degree terms. Introducting the modified Stokes' function (equation 21), the second integral on the right hand side of (30) can be written as:

$$(32)\quad \frac{Re^2}{4\pi\gamma}\int_{\psi_0}^{\pi}\int_0^{2\pi} \Delta g'\, S(\psi)\, d\sigma = \frac{Re^2}{2\gamma}\sum_{n=0}^{\infty} \Delta g_n'\, Q_n$$

Combining (31) and (32) we can write (30) as:

$$(33)\quad \frac{Re^2}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} \Delta g'\, S(\psi)\, d\sigma = \frac{e^2}{2\gamma}\sum_{n=0}^{\infty}\left[X_n - Q_n\right] \; ; \; X_n = \begin{cases} 0 & \text{if } n<2 \\ \frac{2}{(n-1)}, & n \geq 2 \end{cases}$$

$$\sum_{m=0}^{n} (G_{nm}\cos m\lambda + H_{nm}\sin m\lambda)\, P_{nm}(\sin\bar{\varphi})$$

We now can write our final result by re-writing (29) with (33) and (21):

$$(34)\quad N_E = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} \Delta g\, S(\psi)\, d\sigma$$

$$+ \frac{kM}{2 r_E \gamma}\sum_{n=2}^{\infty} Q_n (n-1)\left(\frac{a}{r_E}\right)^n \sum_{m=0}^{n} (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\, P_{nm}(\sin\bar{\varphi})$$

$$+ \frac{e^2}{2\gamma}\sum_{n=0}^{\infty}(Q_n - X_n) \sum_{m=0}^{n} (G_{nm}\cos m\lambda + H_{nm}\sin m\lambda) \;.\; P_{nm}(\sin\bar{\varphi})$$

$$+ e^2\left(\frac{1}{4} - \frac{3}{4}\sin^2\varphi\right) N$$

The first two terms in (34) represent the computation of the undulation with certain approximations while the latter two terms represent the corrections needed to fully refer the solution to an ellipsoidal reference surface. Thus we would design our computations in the following usage:

$$(35)\quad N_E = N_1 + N_2 + \Delta N_1 + \Delta N$$

where $N_1 + N_2$ represents the first two terms in (34) and $\Delta N_1 + \Delta N$ represent the ellipsoidal correction terms. In a later part of the paper we will discuss the numerical values of $\Delta N_1 + \Delta N$.

We can look at two special cases of the above equations. First let $\psi_0 = 0^\circ$ which implies that no gravity anomalies are used in the computation. In this case $Q_n = 2/(n-1)(n > 2)$ so that (34) becomes:

$$(36)\quad N_E = \frac{kM}{r_E \gamma} \sum_{n=2}^{\infty} \left(\frac{a}{r_E}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm}(\sin \bar{\phi})$$
$$+ e^2 \left(\frac{1}{4} - \frac{3}{4} \sin^2\varphi\right) N$$

Equation (36) is the same as:

$$(37)\quad N_E = \frac{kM}{r_E \gamma_E} \sum_{n=2}^{\infty} \left(\frac{a}{r_E}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm}(\sin \bar{\phi})$$

since (Moritz, 1980, eq. (39-17)):

$$(38)\quad \gamma_E = \gamma\left(1 - \frac{1}{4} e^2 + \frac{3}{4} e^2 \sin^2\varphi\right)$$

Equation (37) is the same as given in Rapp (1967, eq. 7) for the computation of a "rigerous" geoid undulation. We thus see the satisfactory reduction of the general case derived in this paper to the special case previously known.

The second special case to consider is when $\psi_0 = 180^\circ$. Then $Q_n = 0$ so that (34) becomes:

$$(39)\quad N_E = \frac{R}{4\pi\gamma} \int_0^{\pi} \int_0^{2\pi} \Delta g\, S(\psi)\, d\sigma$$
$$- \frac{e^2}{\gamma} \sum_{n=2}^{\infty} \frac{1}{(n-1)} \sum_{m=0}^{n} (G_{nm} \cos m\lambda + H_{nm} \sin m\lambda)\, P_{nm}(\sin \bar{\phi})$$
$$+ e^2 \left(\frac{1}{4} - \frac{3}{4} \sin^2\varphi\right) N$$

This result can be compared to the solution given by Lelganam (1970, eq. (3-6)) where numerical tests show that the two formulations yield the same corrections to about ± 1.6 cm. We thus have an additional special case confirmation of our general formula.

3. Ellipsoidal Corrections Using the Modified Stokes' Equation

Tests described by Rapp (1980) and Jekeli (1980) indicated a significant improvement in geoid undulation determinations if the Stokes' function is modified by subtracting $S(\cos\psi_0) = S_0$ from $S(\psi)$. This procedure can be represented in our case by re-writing equation (13) in the following form:

$$(40)\quad N_\varepsilon = \frac{R}{4\pi\gamma}\iint_{\sigma_c} (\Delta g - e^2\Delta g')\,(S(\psi) - S_0)\,d\sigma$$

$$+ \frac{R}{4\pi\gamma}\iint_{\sigma_c} (\Delta g - e^2\Delta g')\,S_0\,d\sigma$$

$$+ \frac{R}{4\pi\gamma}\iint_{\sigma-\sigma_c} (\Delta g - e^2\Delta g')\,S(\psi)\,d\sigma$$

$$+ \Delta N$$

Using a procedure followed before considering Jekeli (1980, sections 2 and 3) we arrive at a result similar to equation (34). Specifically:

$$(41)\quad N_\varepsilon = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} \Delta g\,(S(\psi) - S_0)\,d\sigma$$

$$+ \frac{kM}{2r_\varepsilon\gamma}\sum_{n=2}^{\infty}\bar{Q}_n\,(n-1)\left(\frac{a}{r_\varepsilon}\right)^n \sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\,P_{nm}(\sin\bar{\varphi})$$

$$+ \frac{e^2}{2\gamma}\sum_{n=0}^{\infty}(\bar{Q}_n - X_n)\sum_{m=0}^{n}(G_{nm}\cos m\lambda + H_{nm}\sin m\lambda)\,.\;P_{nm}(\sin\bar{\varphi})$$

$$+ \Delta N$$

We have (Jekeli, 1980, eq. 65):

$$(42)\qquad \bar{Q}_n(\psi_0) = Q_n(\psi_0) + \frac{S(\psi_0)}{(n-1)}\,(P_{n-1}(\cos\psi_0) - \cos\psi_0\,P_n(\cos\psi_0))\;;\quad n \geq 1$$

Equation (41) is the same as (34) with two exceptions: $S(\psi) - S_0$ replaces $S(\psi)$ and $\bar{Q}_n$ replaces Q_n. Numerical tests of both equations will be described in the following section.

4. Numerical Results

We now will evaluate the ellipsoidal correction terms ΔN_1 and ΔN defined in equation (35) and (34) and the similar terms in equation (41). Our starting potential coefficients are those of GEM9 (Lerch et al, 1979) taken to degree 20. The first step in the computation is to find the A_{nm}, B_{nm} coefficients

using equation (13).

We next determined $\Delta N_1 + \Delta N$ as given in equation (35). This is the ellipticity correction to be added to a geoid undulation computed from the first two terms on the right hand side of equation (34). This evaluation was done for $\psi = 10°$, $20°$, and $180°$ and the results are shown in Figures 1, 2, and 3. The maximum correction and root mean square correction for each of of these cases is: (-26 cm, ±6 cm, $\psi = 10°$); (-33 cm, ±10 cm, $\psi = 20°$); (-59 cm, ±18 cm, $\psi = 180°$).

Simiiar computation were carried out when using the modified Stokes' equation. These results are shown for $\psi = 10°$ in Figure 4 and for $\psi = 20°$ in Figure 5. The maximum and the RMS correction for $\psi = 10°$ is (-21 cm, ±5 cm) and for $\psi = 20°$ it is (-27 cm, ±6 cm). Examination of the corresponding figures indicate that the correction for the modified Stokes' integral are somewhat smaller overall than the case with the regular Stokes' function.

The corrections are generally small and below the current accuracy of the data with caps of $10°$ or $20°$. However as more precise computations are carried out in the future, these corrections should be taken into account.

5. The Zero and First Degree Correction Problem

In carrying out the derivation of several of the previous equations, summations were started from 2 instead of 0 by convention or because the Stokes' equation removes zero and first degree terms in a global integration. However the use of the relationships in equation (8) does introduce zero and first degree terms that need to be considered. This problem has been discussed by Leigemann (1970) who assumed the following form of the disturbing potential:

$$
\begin{aligned}
(43)\quad T(r,\bar{\theta},\lambda) = {} & \frac{1}{r^3}\left[A_{2,0}R_{2,0}(\bar{\theta},\lambda) + A_{2,2}R_{2,2}(\bar{\theta},\lambda) + B_{2,2}S_{2,2}(\bar{\theta},\lambda)\right] \\
& + \frac{1}{r^4}\left[A_{3,0}R_{3,0}(\bar{\theta},\lambda) + A_{3,1}R_{3,1}(\bar{\theta},\lambda) + A_{3,2}R_{3,2}(\bar{\theta},\lambda)\right. \\
& + A_{3,3}R_{3,3}(\bar{\theta},\lambda) + B_{3,1}R_{3,1}(\bar{\theta},\lambda) + B_{3,2}S_{3,2}(\bar{\theta},\lambda) \\
& \left. + B_{3,3}S_{3,3}(\bar{\theta},\lambda)\right] + \frac{1}{r^5}\left[A_{4,1}P_{4,1}(\bar{\theta},\lambda)\right]
\end{aligned}
$$

where: $\bar{\theta} = 90° - \bar{\varphi}$

$R_{nm} = \cos m\lambda P_{nm}$

$S_{nm} = \sin m\lambda P_{nm}$

and $\begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} = kMa^n \begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix}$

Note that the A_{nm}, B_{nm} are <u>not</u> the same as given in equation (2). Equation (43) represents a reasonable low degree model of the disturbing potential but it is not meant to be a complete model.

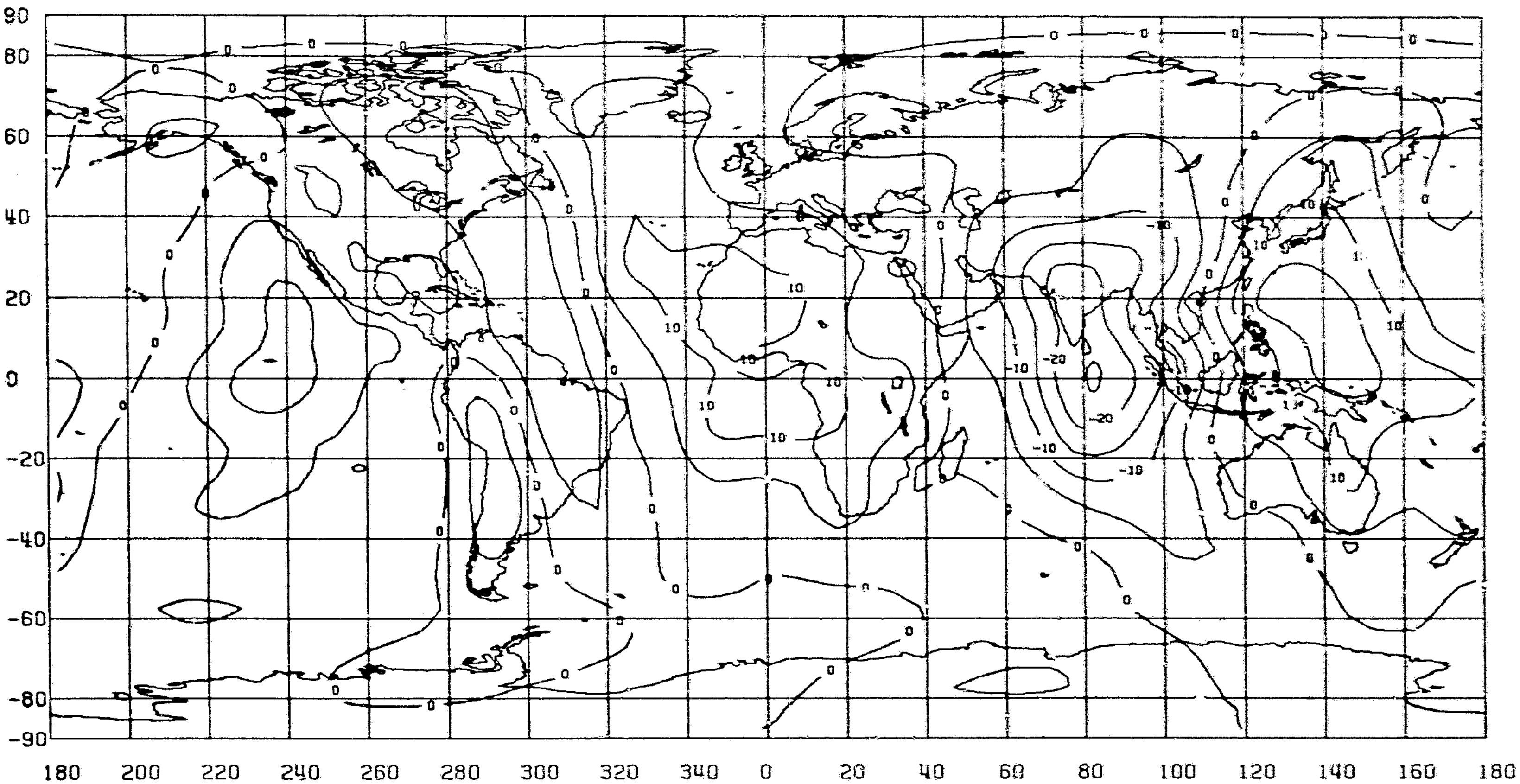

Figure 1

Undulation Correction When Using Anomalies Within a Cap of $\psi = 10^\circ$ with the Regular Stokes' Formulation (Contour Interval = 5 cm)

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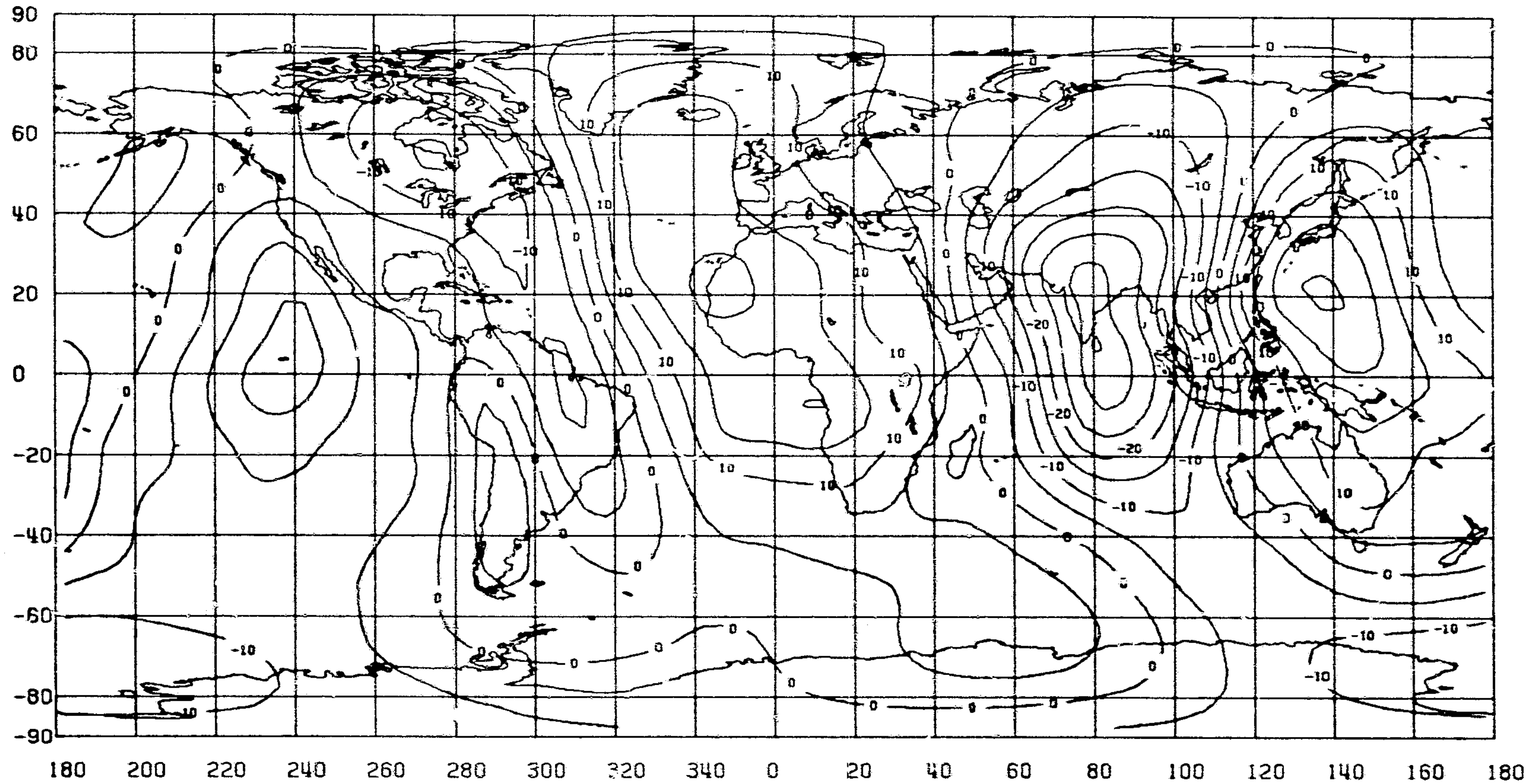

Figure 2

Undulation Correction When Using Anomalies Within A Cap of $\psi = 20°$ with the Regular Stokes' Formulation (Contour Interval = 5 cm)

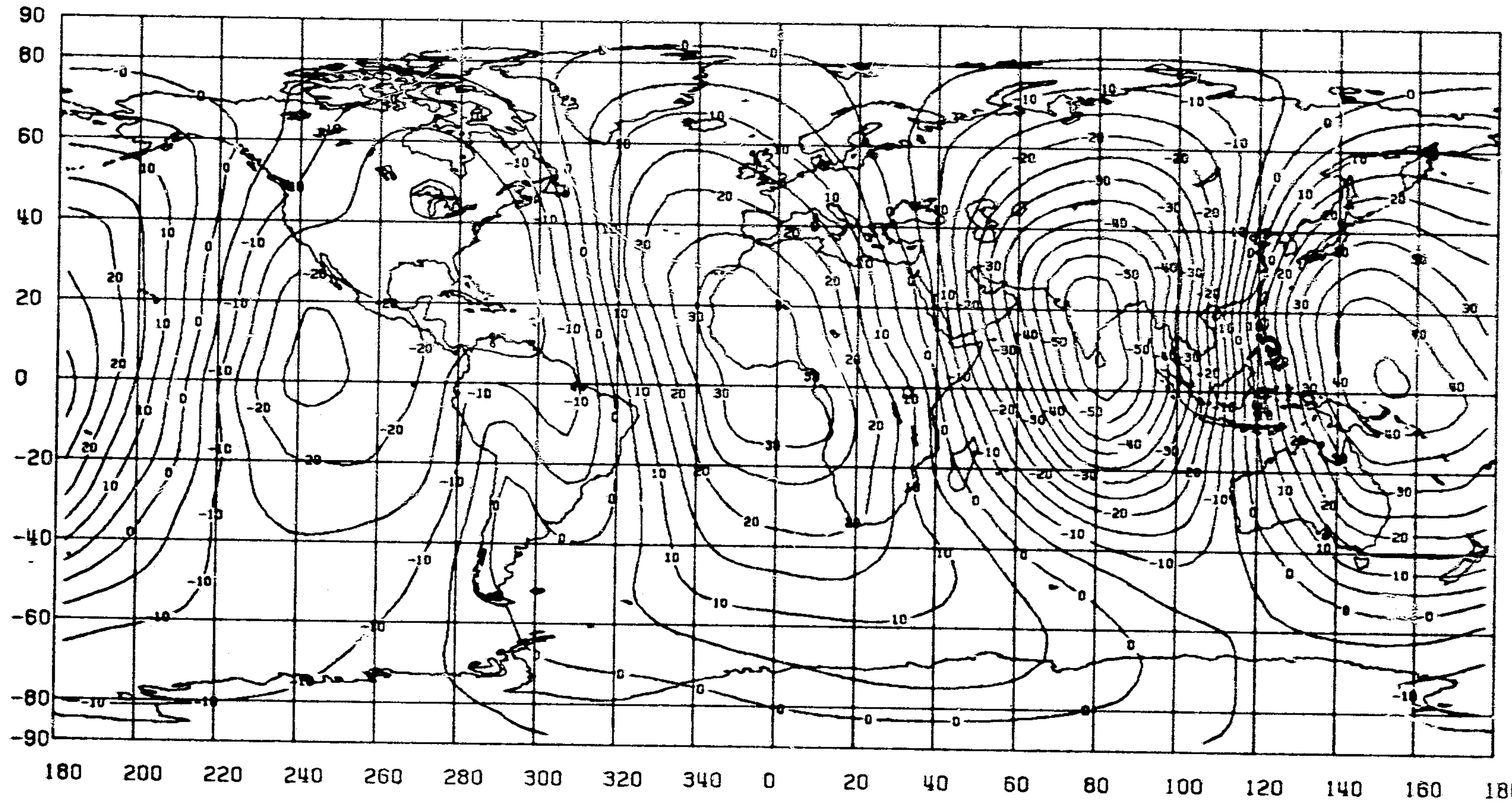

Figure 3

Undulation Correction When A Global Set of Anomalies are Used, $\psi = 180^\circ$; (Contour Interval = 5 cm)

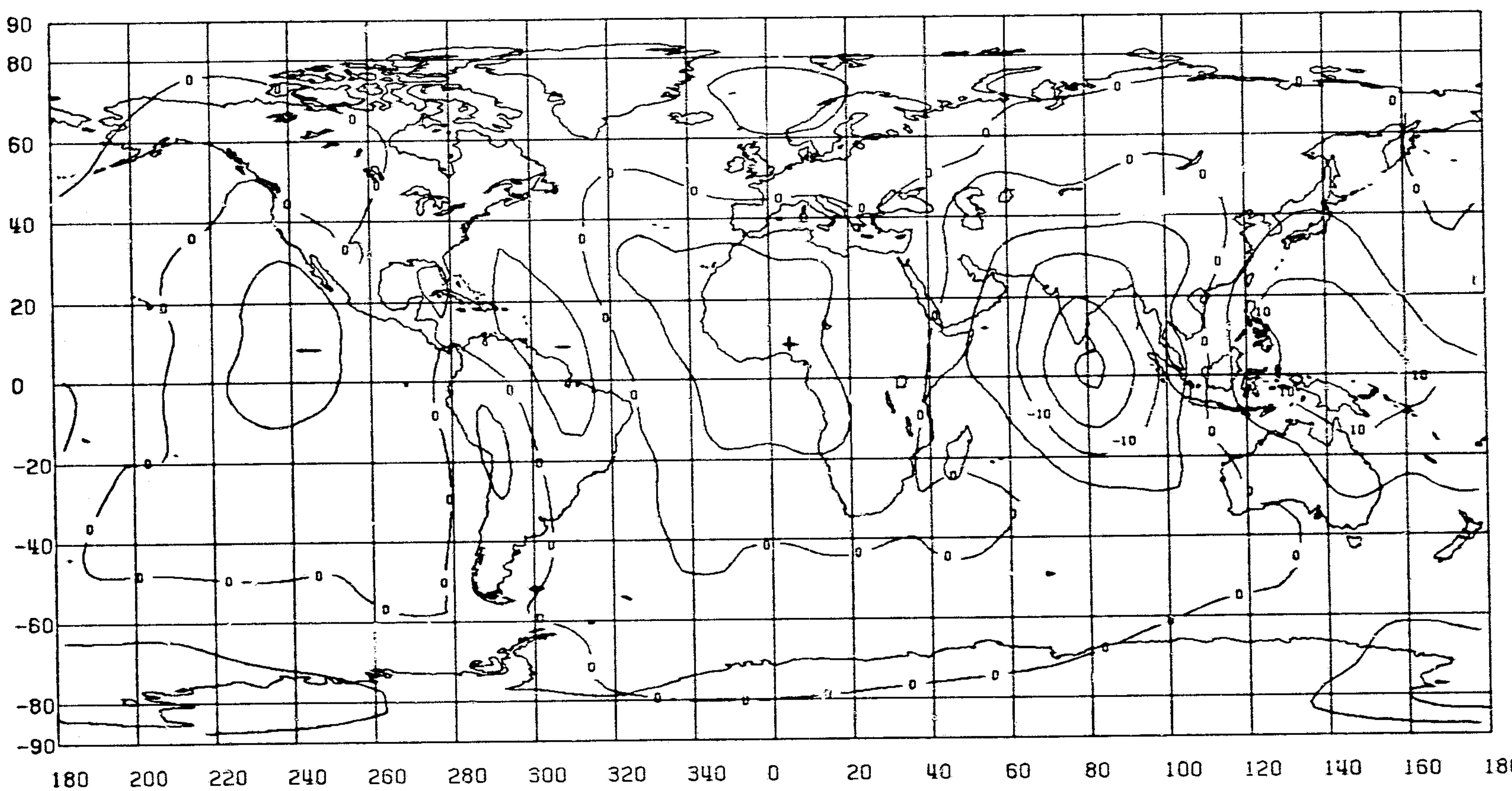

Figure 4

Undulation Correction When Using Anomalies Within a Cap of $\psi = 10^\circ$ with the Modified Stokes' Formulation (Contour Interval = 5 cm)

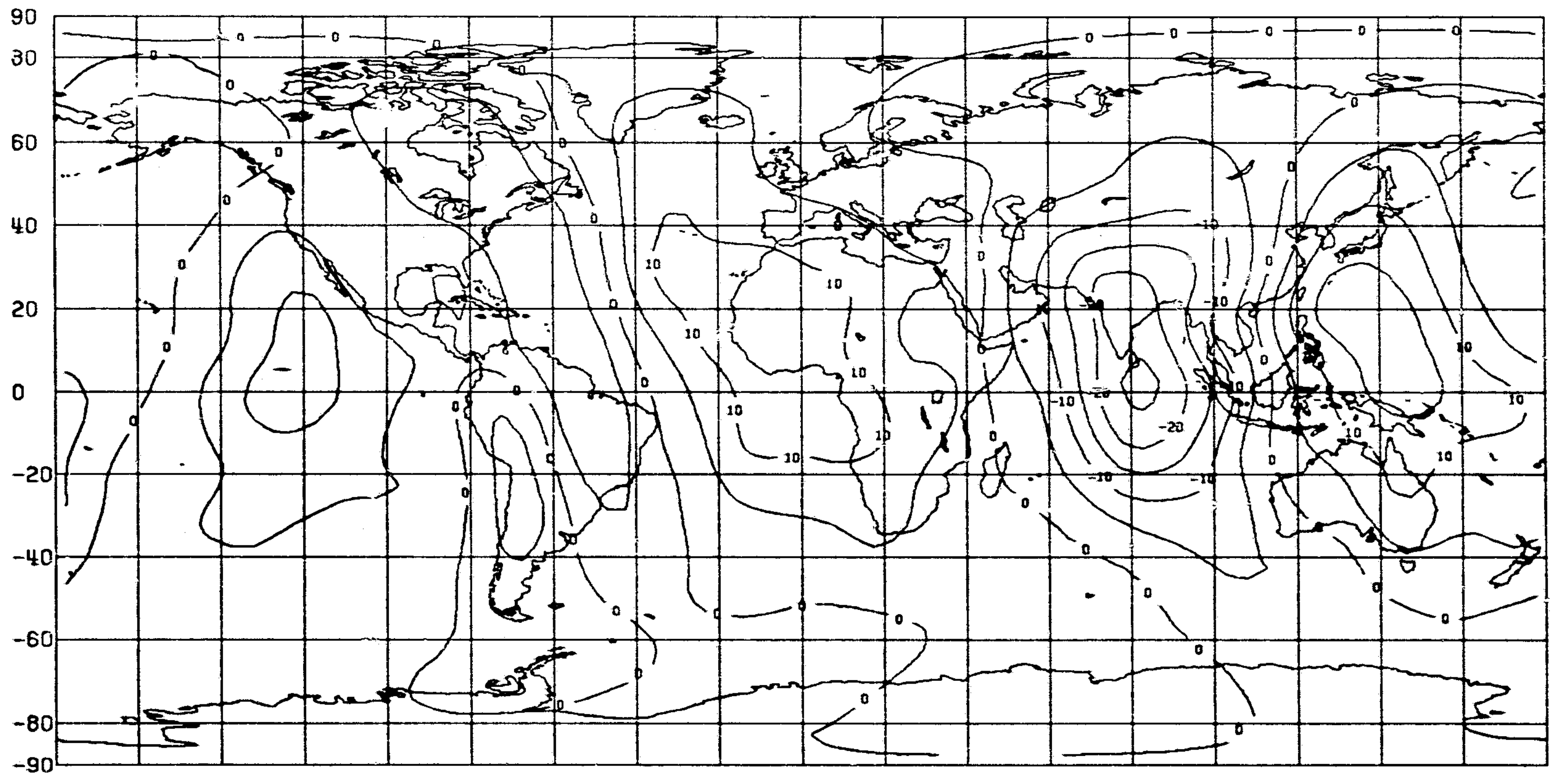

Figure 5

Undulation Correction When Using Anomalies Within a Cap of $\psi = 20°$ with the Modified Stokes' Formulation (Contour Interval = 5 cm)

Lelgemann (1970) shows that the error introduced by the neglect of the zero and first degree terms when developing the ellipticity correction for the global Stokes' integration ($\psi_0 = 180°$) is:

$$(44)\quad \Delta N_{0,1} = \frac{e'^2}{\gamma}\left[-\frac{1}{5}a_{2,0} - \frac{12}{35}a_{3,0}R_{1,0} - \frac{24}{35}(a_{3,1}R_{11} + b_{3,1}S_{11})\right]$$

where:

$$\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix} = \frac{kM}{a}\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix}$$

Using the coefficients of GEM9 we have evaluated equation (44) with the results shown in Figure 6. As is obvious from (44) this correction is very long wavelength. The magnitude is quite small with the maximum correction being ± 7 cm. We would expect that the correction for the small cap sizes used in practice would be considerably smaller than this as was seen for the usual ellipticity correction. Therefore we will not pursue the derivation of this correction term for the cap case.

6. Ellipticity Corrections for the Marsh-Chang Geoid

For the past several years Marsh and Chang (1976, 1978) have computed detailed geoid undulations combining potential coefficient information and terrestrial gravity data. The method used by them is called Method A in Rapp and Rummel (1975) or Method 1 in Rummel and Rapp (1976). The specific equations used by Marsh and Chang in their recent papers are as follows:

$$(45)\quad N(\varphi,\lambda) = N_1 + N_2$$

where:

$$(46)\quad N_1 = R\sum_{n=2}^{\bar{n}}\sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\,P_{nm}(\sin\bar{\varphi})$$

$$(47)\quad N_2 = \frac{R}{4\pi\gamma}\iint_{\sigma_c}(\Delta g - \Delta g_s)\,S(\psi)\,d\sigma$$

$$(48)\quad \Delta g_s = \gamma\sum_{n=2}^{\bar{n}}\sum_{m=0}^{n}(n-1)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\,P_{nm}(\sin\bar{\varphi})$$

Here $\bar{n}$ is the maximum degree used with the potential coefficients. In practice the integration in the cap has taken place using just $1° \times 1°$ anomalies or these anomalies in conjunction with smaller block sizes such as $5' \times 5'$, and $15' \times 15'$. The integration cap has been $10°$, $20°$ or presumably $0°$ when insufficient gravity data is present. In view of our previous discussions we now are interested in the geoid error caused by the spherical approximation in equation (46) and (47).

For convenience we introduce A_n as follows:

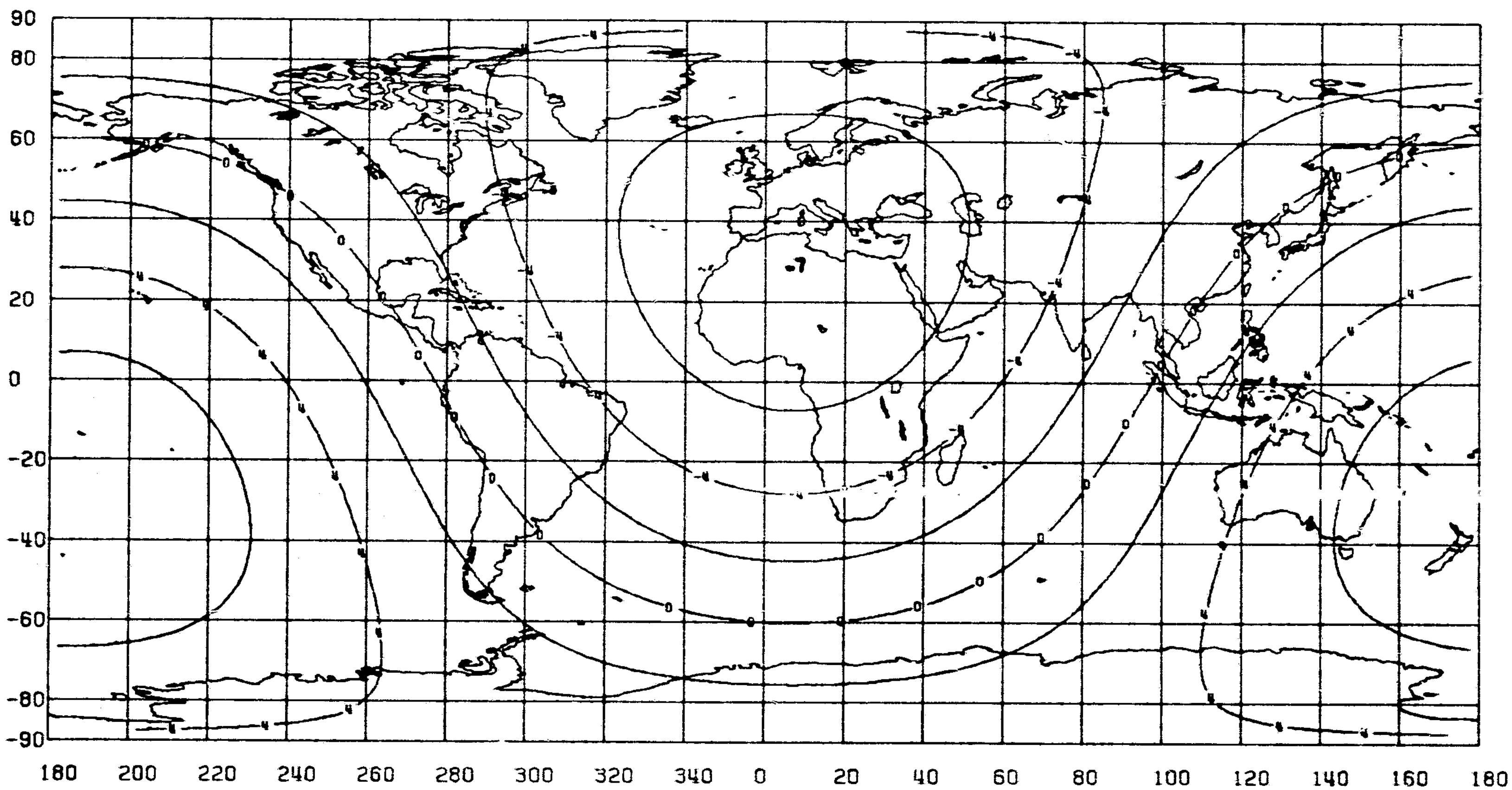

Figure 6

Zero and First Degree Ellipticity Corrections
for 180° Cap (Contour Interval = 2 cm)

(44) $A_n = \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm} (\sin\phi)$

The ellipsoidal error in the Marsh/Chang geoid will be:

(50) $\Delta N_{M/C} = N_E \text{ (eq. 34)} - N \text{(eq. 45)}$

In order to reduce (50) the following equality can be used

$$(51) \qquad \frac{R}{4\pi\gamma} \iint_{\sigma_c} \Delta g_e\, S(\psi)\, d\sigma = \frac{R}{2} \sum_{n=2}^{\bar{n}} (2 - Q_n (n-1))\, A_n$$

An erroneous form of (51) was given in Rapp and Rummel (1975, eq. 33). Using (34), (45), and (51), equation (50) can be written as:

$$\begin{aligned}(52)\quad \Delta N_{M/C} = {} & \frac{kM}{2 r_E \gamma} \sum_{n=2}^{\bar{n}} Q_n (n-1) \left(\frac{a}{r_E}\right)^n A_n \\ & - \frac{R}{2} \sum_{n=2}^{\bar{n}} Q_n (n-1)\, A_n \\ & + \frac{e^2}{2} \sum_{n=0}^{\bar{n}} (Q_n - X_n) \sum_{m=0}^{n} (G_{nm} \cos m\lambda \\ & + H_{nm} \sin m\lambda)\, P_{nm} (\sin\phi) + \Delta N\end{aligned}$$

We have evaluated (52) with the GEM9 potential coefficients ($\bar{n} = 20$) for $\psi = 0°$, $10°$, and $20°$. These results are shown in Figure 7, 8, and 9. The maximum correction and root mean square correction for each of these cases is (101 cm, ±27 cm, $\psi = 0°$), (44 cm, ±16 cm, $\psi = 10°$), (-45 cm, ±14 cm, $\psi = 20°$).

7. Summary and Conclusions

This paper has developed the formulas needed to compute the correction for geoid undulation computations made from the combination of potential coefficient information and terrestrial gravity data. The first procedure developed the formulas needed for the precise computation of the geoid considering the ellipsoid as a reference surface and using the usual Stokes' equation. The corrections are a function of the cap within which gravity data is used. For a cap size of 20° the maximum correction was -35 cm.

Another case was considered with the use of the modified Stokes' function In this case the maximum correction for $\psi = 20°$ was -27 cm.

The third case considered was that for the ellipticity corrections for the Marsh/Chang geoid. If a cap of 0° was used the maximum correction was 101 cm; if the cap was 20°, the maximum correction was -45 cm.

All the corrections have been computed using the GEM9 potential coef-

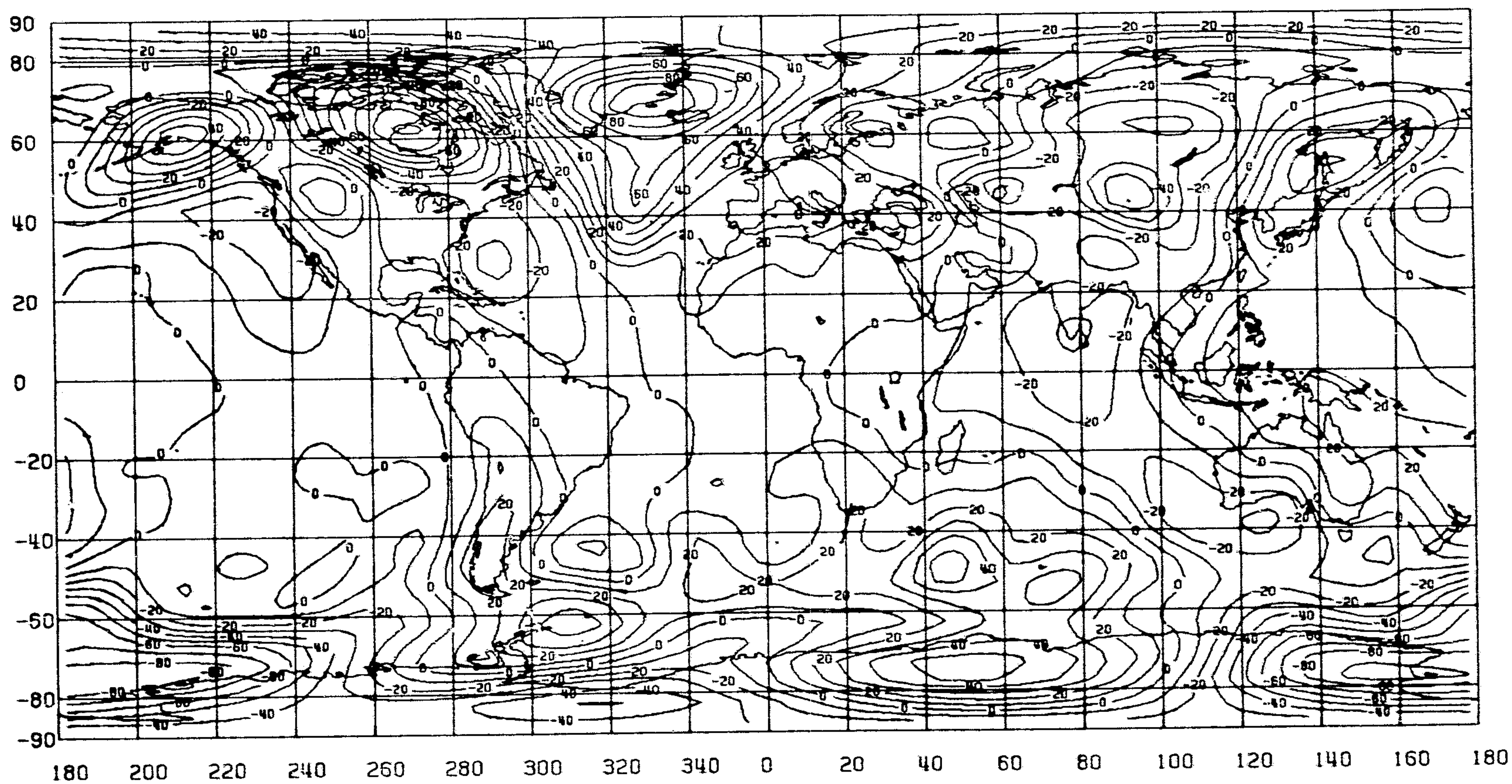

Figure 7

Ellipticity Correction for Marsh/Chang Geoid Undulation
if $\psi = 0°$, (Contour Interval = 10 cm)

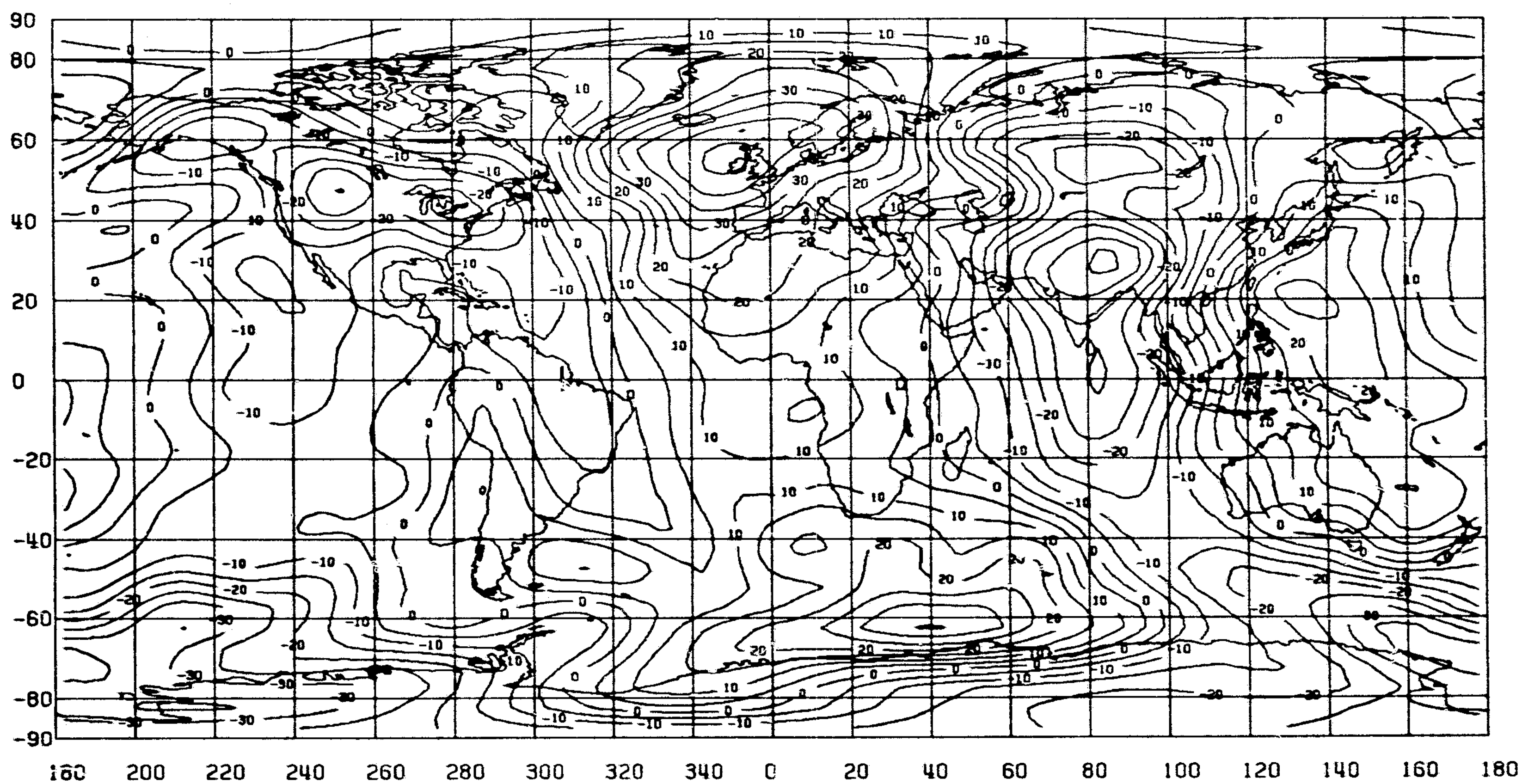

Figure 8

Ellipticity Correction for Marsh/Chang Geoid Undulations
if $\psi = 10°$, (Contour Interval = 5 cm)

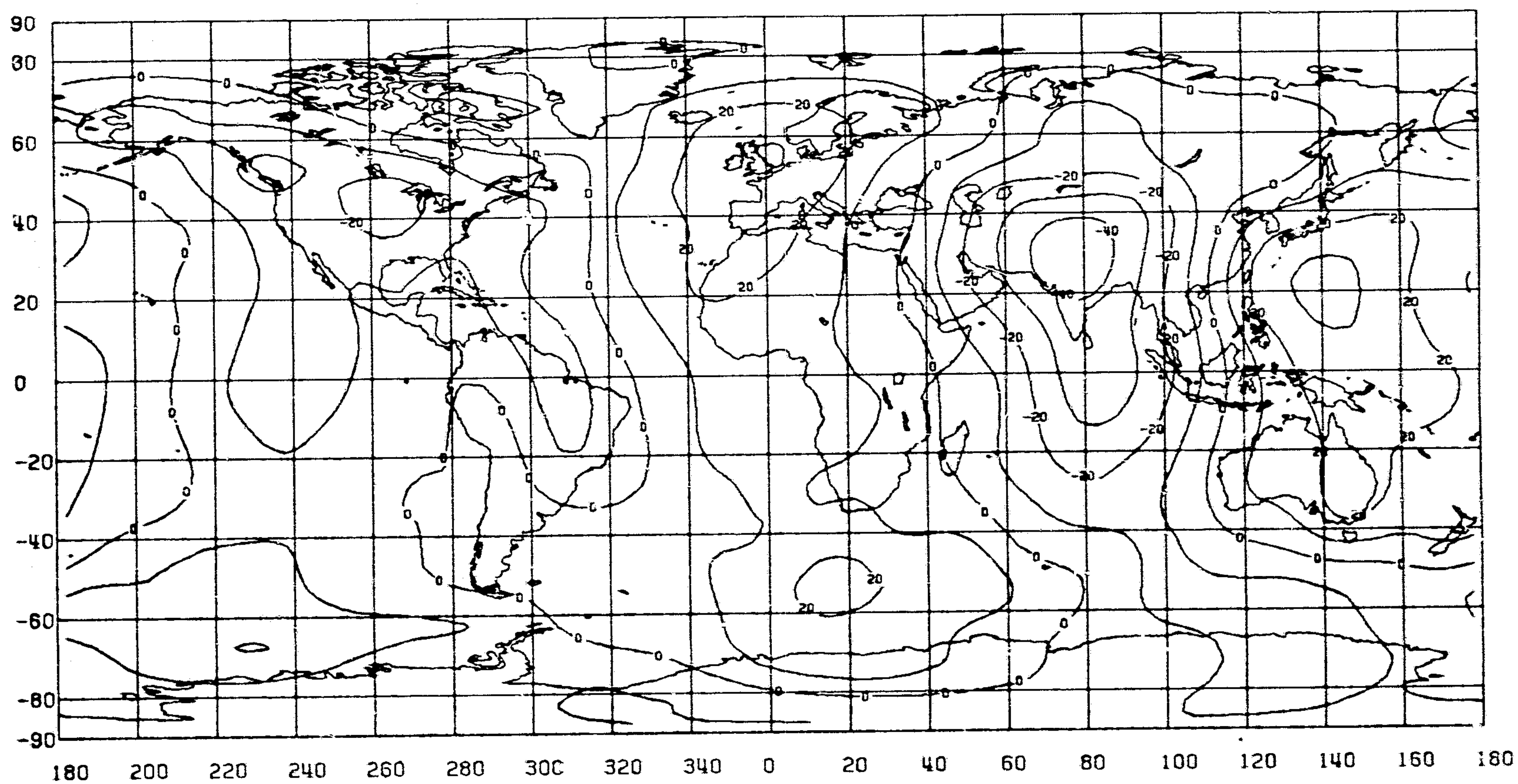

Figure 9

Ellipticity Correction for Marsh/Chang Geoid Undulations
if $\psi = 20°$, (Contour Interval = 10 cm)

ficients taken to degree 20. Errors in these coefficients or the use of additional higher degree terms should not significantly effect these results.

As can be seen from the various maps, these correction terms are fairly long wavelength. Therefore in some applications working with altimeter data and gravimetric geoids, the correction could appear as a constant difference. In some cases, for example, in examining the difference between the sea surface and the gravimetric geoid of Marsh/Chang a net correction, across the Pacific Ocean, of about 35 cm should be made. If, in the future, we are to determine highly accurate geoids from potential coefficients and terrestrial gravity data, the corrections or problem formulation given in this paper should be used.

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